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ORIGINAL ARTICLE

Numerical study of partial differential equations to estimate thermoregulation in human dermal regions for temperature dependent thermal conductivity

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Abstract The paper deals with the temperature distribution in multi-layered human skin and subcutaneous tissues (SST). The model suggests the solution of parabolic heat equation together with the boundary conditions for the temperature distribution in SST by assuming the thermal conductivity as a function of temperature.

The model formulation is based on singular non-linear boundary value problem and has been solved using finite difference method. The numerical results were found similar to clinical and computational results.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 92BXX; 92CXX; 92C35; 92C50; 46N60

1. Introduction

The unstable ambient temperature plays an important role for the disturbance in human thermoregulatory system. The effect of surrounding temperature makes its way via dermal layers and leads to hyperthermia and hypothermia to the body core and tissue necrosis to the body peripherals. Several researchers studied the distribution of temperature in the human body organs in relation to several environment temperatures. The mathematical model for temperature distribution in the human dermal layers can be represented as a boundary value problem. In this study, the domain is consisting of dermal layers and the formulation is based on the differential equation of heat conduction as

$$k\frac{d^2T}{d^2r}+\frac{2k}{r}\frac{dT}{dr}+Q=0 \tag{1}$$

and the boundary conditions are

$$\lim_{r\to0^+}\frac{dT}{dr}=0,\quad \text{and}\quad -k\left(\frac{dT}{dr}\right)_{r=R}=E(T_H-T_a) \tag{2}$$

where r is the radial distance from the core of the domain, R – the radius of the domain, E – the evaporation term, T_a – the ambient temperature, T_H – the periphery temperature,

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Q – the heat production per unit volume and k is the thermal conductivity at the dermal regions.

Various mathematical models were formulated to study the effect of environmental temperatures on human dermal regions. Our group had also developed numerous models in this direction. Few models are demonstrated by Khanday and Saxena [1–3], but the thermal conductivity was assumed as constant or function of space parameter to describe the thermoregulation in biological tissues. The main purpose of this study is to estimate the temperature profiles at the dermal regions with respect to changes in ambient temperature and thermal conductivity as a function of temperature.

Thron [4] studied the above model to estimate the temperature distribution in human head and suggested that if there is no singularity in the above differential equation, then the solution is given by

$$T(r) = T_a + \frac{QR^2}{6k}\left[1 + \frac{2k}{ER} - \left(\frac{r}{R}\right)^2\right] \tag{3}$$

In addition, he calculated the temperature distribution by assuming additional heat sources while the cooling of blood at periphery by the Eq. (3) with

$$Q = Q_0 + Q_b \tag{4}$$

where $Q_b = Vs(T_1 - T)$, Q_0 is the heat production of tissue, V is volume of the flow of blood in unit time, T_1 is the deep temperature of the core and s = 0.9 cal/°C cm^3.

Richardson and Whitelaw [5] predicted the temperature profiles in the biological tissues by keeping skin surface as functions of temperature. Flesch [6] estimated the temperature distribution using the heat Eq. (1) by assuming a heat generation rate as an explicit function of the radial distance and an implicit function of the environment temperature. Khanday and Saxena [2] calculated the mass and temperature distribution at multi-layered skin and sub-dermal tissues by using variational finite element method with respect to various environmental temperatures. Also they studied the thermostat phenomenon of brain tissue and estimated the cold stress at multi-layered human head with respect to ambient temperatures.

The present work is an attempt to study the distribution of temperature at deep dermal layers for heterogeneous thermal conductivity as a function of temperature.

2. Mathematical formulation of the model

The mathematical modelling for the estimation of temperature distribution in human body has gained interest among many researchers and our group has published many papers by taking into account various parameters and other physiological aspect of the domain under study. The heat transfer in biological tissues was considered initially by Pennes [7] and further elaborated by other researchers as well. The role of temperature to the changes in thermal conductivity of the material has been incorporated by means of the term $k(T) = k_0(T - T_H)^n$. The mathematical model of the heat transfer in the human dermal regions has been considered by the following usual differential equation of heat conduction

$$\rho c\frac{\partial T}{\partial t} = k(T)\nabla^2 T + k'(T)\nabla \cdot T + Q \tag{5}$$

where ∇ is an operator determining the first order partial derivative of T with respect to three dimensional systems; ρ, c and k represent the density, specific heat of the tissue and thermal conductivity respectively.

In case of steady state processes, the above equation can be written as

$$k_0\frac{d^2T}{dr^2} + \frac{2k_0}{r}\frac{dT}{dr} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{Q}{(T - T_H)^n} = 0 \tag{6}$$

Using the fact that the heat generation term Q is a function of temperature T. Therefore, let $Q = q_1(37 - T)$ for some positive constant q_1.

Thus, singular boundary value problem determining the heat conduction at dermal layers has been established as follows

$$k_0\frac{d^2T}{dr^2} + \frac{2k_0}{r}\frac{dT}{dr} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{q_1(37 - T)}{(T - T_H)^n} = 0 \tag{7}$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \quad T(R) = T_H \tag{8}$$

where Eq. (8) is the boundary conditions.

In order to non-dimensionalize the above boundary value problem, we make use of the following transformations

$$y = T - T_H \quad \text{and } t = r/R,\ 0 < t < 1, \tag{9}$$

The following system of equations results from the transformation

$$k_0\frac{d^2y}{dt^2} + \frac{2k_0}{t}\frac{dy}{dt} + \frac{nk_0}{y}\left(\frac{dy}{dt}\right)^2 + \frac{q_1(37 - y - T_H)R^2}{y^n} = 0,$$
$$0 < t < 1, \left.\frac{dy}{dt}\right|_{t=0} = 0, \quad y(1) = 0 \tag{10}$$

Define the following substitutions,

$$c(t) = t^2 \quad \text{and } f(t, y, cy') = \frac{nk_0}{y}\left(\frac{dy}{dt}\right)^2 + \frac{q_1(37 - y - T_H)R^2}{y^n}$$

we have,

$$\begin{aligned}&\frac{1}{c(t)}[c(t)y'(t)]' + f(t, y, cy') = 0\\&y'(0) = 0, \quad y(1) = 0\end{aligned} \tag{11}$$

The solution of the singular non-linear boundary value problem (11) exists and is unique as discussed by Celik [8]. The solution was approximated by means of the finite difference method.

3. Solution and interpretation of the model

The distribution of temperature in human dermal regions can be sought to solve the boundary value problem (10) numerically.

The finite difference method will be applied to solve (10) as follows:

Dividing (0, 1) into p subintervals with the length of each subinterval as $h = 1/p$, then by the central differences, the above equation for $i = 0$, can be written as

$$2k_0y_1 + \frac{q_1(37 - y_0 - T_H)R^2h^2}{y_0^n} - 2k_0y_0 = 0 \tag{12}$$

and for $i = 1, 2, 3, \ldots, p - 1$

$$\left(1+\frac{1}{i}\right)y_{i+1}+nk_0\frac{(y_{i+1}-y_{i-1})^2}{4y_i}+\frac{q_1(37-y_i-T_H)R^2h^2}{y^n}$$
$$-2k_0y_i+\left(1-\frac{1}{i}\right)y_{i-1}=0 \quad (13)$$

4. Numerical computation

The resulting non-linear systems of Eq. (13) were solved numerically. The temperature at dermal regions was estimated with respect to various environmental temperatures.

For $p = 1/3$, $q_1 = 0.000002T_H^2$ for forehead $T_H = 33.03 + 0.14(T_a - 10)$, $k_0 = 0.00009T_H(37 - T_H)^{-1/3}$, $m = 100$ and $T_a = 0, 10, 21, 25$ °C.

The change in tissue temperature at various dermal layers with respect to radial distance is demonstrated in Fig. 1. The figure reveals that there are gradual changes of temperature with radial distances. The numerical solution of the Eq. (13) together with the boundary conditions has been found for the ambient temperatures $T_a = 0$ °C, 10 °C and are interpreted in Fig. 1. The study in this paper shows same temperature variation of tissues that of Thron [4] with few significant changes in some results.

The main reason for such differences is that Thron [4] treated the thermal conductivity as constant whereas it is temperature dependent in this case. Therefore, it may be said that the present study is much more realistic. The results described in Fig. 2 at ambient temperatures $T_a = 21$ °C and 25 °C were compared with the solution of other researchers including Richardson [5] and Cleik [8]. It is evident from the Figs. 1 and 2 that the temperature variation in radial distances shows the same tendency with existing study.

The model also suggests the heat generation of tissues due to metabolism interpreted in Fig. 3 with respect to radial distance from the centre. The Fig. 3 reveals the fact that the heat within the tissue is increasing continuously from the centre of the domain and the humidity and moisture concentration in the atmosphere is retarding the heat stress. It also shows that the heat generation in these regions increases when the environment temperature decreases. The role of blood mass flow rate for the heat distribution also varies due to changes in environmental temperature due to perfusion property. The effect of blood perfusion is a determinant factor of skin conductivity and temperature as reflected in the illustration of Fig. 3.

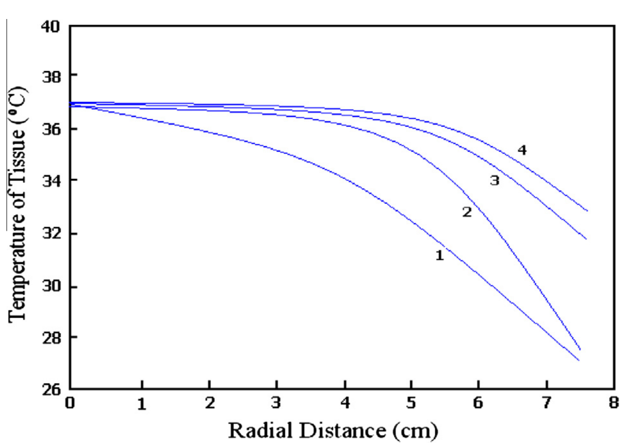

Figure 1 Temperature distribution at the human dermal regions at $T_a = 0$, 10 °C.

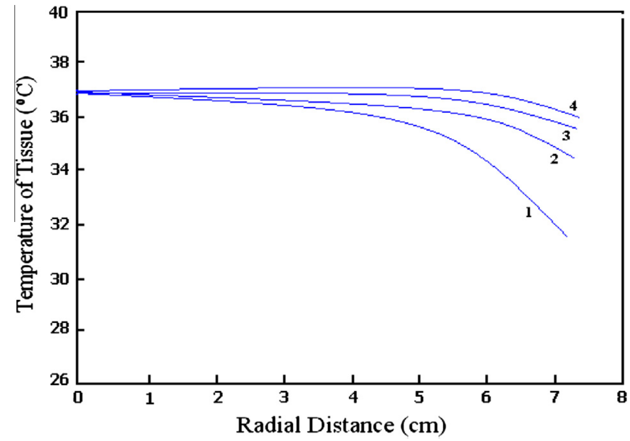

Figure 2 Temperature distribution at the human dermal regions at $T_a = 21$, 25 °C.

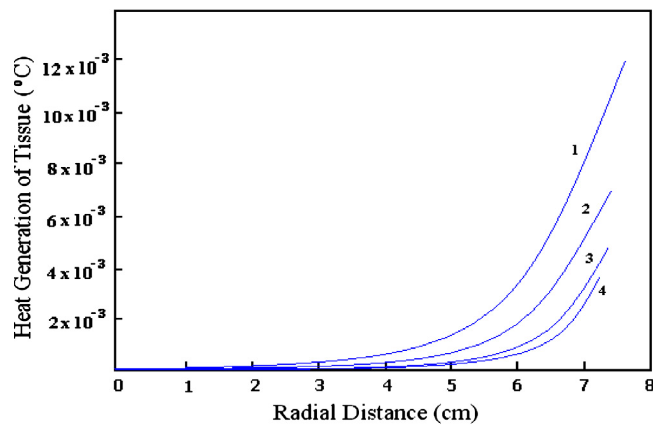

Figure 3 Heat generation at the dermal tissue with respect to radial distance.

5. Discussion and conclusions

A mathematical model describing the heat transport in dermal regions was formulated. As an application to non-linear singular boundary value problem, a mathematical model for the heat conduction in the cutaneous and subcutaneous tissues has been solved by using numerical method. The main aspect of this study reflects some innovation in the existing models by means of variable thermal conductivity. It is evident to say that the empirical models reflect the dependence of thermal conductivity and temperature changes in the material. Thus, the study shows some realistic values for the estimation of thermoregulation in human dermal layers. The value of thermal conductivity gradually increases from outer regions towards core with increase in temperature. In order to be more realistic, k has been considered as a function of T and a non-linear singular differential equation with suitable boundary conditions were invoked in the model. The solution of the model has been carried out by using numerical finite difference method and the results were interpreted in Figs. 1–3.

The estimation of temperature distribution in human head for various ambient temperatures was done by various researchers including Khanday and Saxena [1–3] and Thron [4]. They have realised that sublingual temperature of the head is not affected from the environmental temperature. The results obtained by Jiang [9] to study the effects of thermal properties and geometrical dimensions on skin burn injuries are comparatively similar to that of our results. But the heterogeneous character of thermal conductivity in our study reveals some important and realistic outcome for thermal stability in domain under study. Also it has been said the temperature at the centre of the head regions is not affected by the environment temperature as well by Thron [4]. The similar argument with an improvement in relation with thermal conductivity has shown that temperature varies slowly near epidermal regions and there is a rapid growth of temperature from dermis to hypodermis. The competence of tissues to generate heat at various environmental disturbances has also been studied in the model.

The thermal stress due to severe environmental conditions to describe the degrees of injuries at dermal regions can be further investigated by the scientists from biomedical and biophysical areas. The estimated temperature profiles at the nodal points of the regions shall be helpful to investigate hypothermic and hyperthermic conditions at clinical and other surgical situations.

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